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Error Analysis of Finite Element Results on Plates with Nonuniform Grids

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Abstract

DISCRETE Dirac delta functions in two dimensions and the governing plate differential equations are used to produce continuous approximations from discrete finite-element data on nonuniform grids. The continuous approximation can be differentiated to compute continuous stresses/moments at any point in the plate and, when substituted in the differential equations, provides a residual error. The paper presents application examples of the procedure and studies the use of the "smoothed" solution in a Zienkiewicz-Zhu error estimator to assess the performance of the present analysis. In an example involving the linear response of a clamped square plate under uniform transverse load, the present procedure dramatically corrected and smoothed the discrete finite element results.

Contents

The finite element method is a powerful tool that provides approximate solutions to engineering problems. Hence a reliable estimate of the accuracy of stresses and deflections computed by the method is necessary for efficient and dependable designs. An earlier paper[1] outlined a general approach to the error analysis and correction of results from the finite element method (FEM). The approach suggested using Newton's method for solving the Euler equations of shell theory for components of shell structures. Continuity of stresses and deformations between shell components are enforced by using substructuring.

References 2 and 3 addressed and corrected an oscillatory behavior of the continuous solution between boundary nodes of rectangular plate sections (caused by a Gibbs' phenomenon in the Fourier sine series used in the numerical analysis) and applied the general approach of Ref. 1 to postbuckled, stiffened, rectangular, composite plates with initial imperfections. The continuous smoothed solution obtained from the error analysis[1-3] when substituted back into the governing differential equations yields a "residual error." This residual error provides a measure of how well the smoothed solution, and by extension, the finite element solution, satisfies shell theory.

Earlier work[1-3] was limited to the error analysis of FEM results from uniform rectangular grids. In general, the discretization of structural members for analysis with the finite element method can involve nonuniform grids and is often the case when adaptive refinement is used in the finite element analysis. The present paper extends the approach of earlier work on error analysis of finite element results for postbuckled plates to the general case of nonuniform grids.

As in earlier work, the continuous solution for the transverse deflection W consists of two series solutions W_I and W_E,

$$W = W_I + W_E \tag{1a}$$

The generalized coordinates in each series are computed from discrete data; W_I from data at centroids of elements and W_E from data at nodal points on the plate boundary. The functions W_I and W_E must satisfy the two governing plate partial differential equations

$$L_{33}(W_I) = N_3(F_0,\ W_0) + N_3(F_0, \overline{W}) + q \tag{1b}$$

$$L_{33}(W_E) = 0 \tag{1c}$$

where

$$L_{33}(W) = D_{11}W_{,xxxx} + 2(D_{12} + 2D_{66})W_{,xxyy} + D_{22}W_{,yyyy} \tag{2a}$$

$$N_3(F,\ W) = N_xW_{,xx} + 2N_{xy}W_{,xy} + N_yW_{,yy} \tag{2b}$$

with D_{ij} as the plate bending stiffnesses, and N_x, N_y, N_{xy} the in-plane stress resultants. The zero subscript on the right side of Eq. (1b) denotes discrete data from the finite element solution generally available at the centroids of the quadrilateral elements, $\overline{W}$ denotes initial imperfection, and q is a known distributed surface load.

To compute the particular solution W_I, a continuous representation of N_3 is sought. An imaginary uniform extended grid is attached to the plate and an interpolation function is written as the sum of discrete Dirac delta functions passing through the discrete values of the function (N_3) at the centroids of the nonuniform mesh as

$$N_3(x_k,\ y_k) = \sum_{i=1}^{I} C_iD(\xi_i,\ \eta_i,\ \xi_k,\ \eta_k) \tag{3a}$$

where D is a Dirac delta function in two dimensions defined as

$$D(\xi_i,\ \eta_i,\ \xi_k,\ \eta_k) = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1} \frac{4\sin m\xi_i \sin n\eta_i \sin m\xi_k \sin n\eta_k}{(M+1)(N+1)} \tag{3b}$$

and where ξ and η are extended grid variables,[2] M and N are the number of nodes in the x and y directions, and I is the number of data points. A discrete Dirac delta function is a continuous approximation to a point load when M and N are large.

The interpolation formula is written for each centroidal point x_k, y_k on the nonuniform mesh and then the resulting linear set of equations are solved for the unknown coefficients C_i. Equation (3a) then provides the continuous approximation for N_3 as a function of x and y. Projecting on to the centroids of the attached uniform rectangular grid gives a complete set of data. The continuous solution for W_I is now computed as in earlier work by first using numerical harmonic analysis[3] to compute the coefficients N_{3mn} in a double sine series passing through the interpolated data at the centroids of the uniform grid.

$$N_3(F_0,\ W_0) + q = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1} N_{3mn}\sin m\xi \sin n\eta \tag{4a}$$

The generalized coordinates in the continuous solution W_I are the Fourier coefficients A_{mn} in the series

$$W_I = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1} A_{mn}\sin m\xi \sin n\eta \tag{4b}$$

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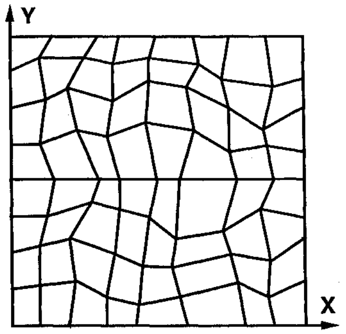

Fig. 1 Clamped plate under transverse load.

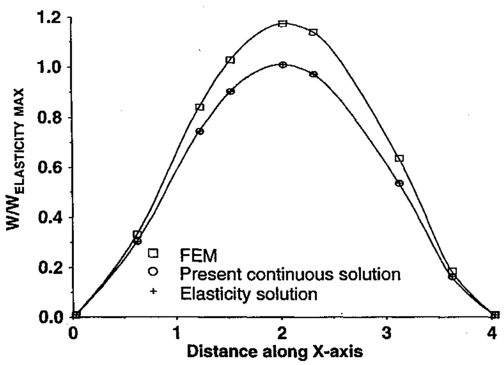

Fig. 2 Transverse deflection W along the line $y/b = 0.5$.

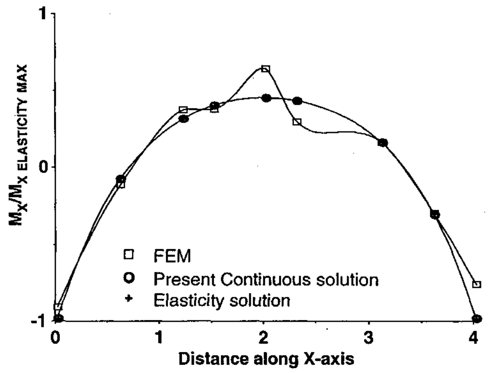

Fig. 3 Moment resultant M_x along the line $y/b = 0.5$.

and are obtained by substituting Eqs. (4a) and (4b) into Eq. (1b) and comparing coefficients of like terms.

The solution W_E is added to W_I to obtain a better fit of the discrete results on the boundary. W_E is assumed as a series that satisfies Eq. (1c) term by term:

$$W_E = \sum_{n=1}^{N} f_n(x) \sin n\eta + \sum_{m=1}^{M} g_m(y) \sin m\xi \tag{5}$$

The discrete boundary conditions are known at nonuniformly located nodes on the boundary. Cubic spline interpolation is used to compute their values at the boundary nodes of the attached uniform grid. Special treatment required to satisfy the boundary conditions at corner nodes is described in Ref. 3.

The procedure for the computation of the continuous solutions for the in-plane displacements U and V is similar to the solution procedure for W. A double Fourier series is used to generate particular solutions U_I and V_I of the in-plane equilibrium equations. All of the quantities on the right-hand sides of the in-plane equilibrium equations are functions of derivatives of W which are known from the continuous solution presented earlier. Hence, the particular solution for the in-plane displacements proceeds as described in Ref. 3 for the uniform grid case.

As in the case of the continuous solution for W, the solutions U_E and V_E are added to U_I and V_I so as to obtain a better fit of the discrete results on the boundary. For the in-plane solution, additional terms[3] U_P and V_P are required to account for rigid-body displacements and rotations.

The stress and moment resultants at any point on the plate can now be computed from the continuous solution for the displacements and their derivatives. A measure of the error in the continuous solution derived from the finite element results is $E_3(x, y)$, the residual error in satisfying the transverse equilibrium equation

$$E_3 = L_{33}(W) - N_3(F, W + \overline{W}) - q \tag{6}$$

Figure 1 shows a fairly distorted mesh used to model a clamped square plate subjected to a uniform transverse pressure load. A closed-form solution exists for this linear plate bending problem.[4] Results from a finite element analysis of this problem were "smoothed" using the present procedure. Figures 2 and 3 compare the displacement W and the moment M_x variation along an axis of symmetry. The FEM result for the deflection W is not in good agreement with the elasticity solution, being about 15% higher. However, there is excellent agreement between the derived continuous approximation and the elasticity solution. In addition, the FEM result for the bending moment M_x is oscillatory in a region at the center of the plate where the second derivatives of W are rapidly varying and are computed inaccurately due to numerical differentiation in the finite element method. Nevertheless, the continuous solution derived from the FEM results using the present analysis removes the oscillations and agrees well with the elasticity solution. The use of the "smoothed" solution in a Zienkiewicz-Zhu error estimator is discussed in the main paper.

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